Série 2b Solutions

Exercise 2b.1 – Poisson effect

Considering the statue seen in Figure 2b.1, we want to study the lateral strain of the column holding it. The statue's mass is 883 kg and its weight is uniformly spread on the column's cross-section. The material of the column is brittle. The Young's modulus of the material is 1.10 GPa. Its Poisson's ratio is 0.34. The height of the column is 2.1 meter in the two cases.

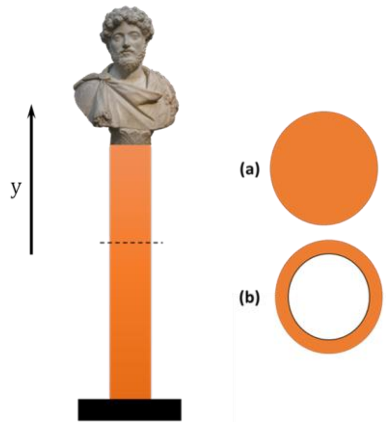

Figure 2b.1 | Schematic of the column with the statue on top and the two different types of cross - sectional areas.

Between cross-section a and b, which is the best solution to minimize the lateral strain to which the column is submitted? Verify your assumption by calculating the lateral strain of the column for the two following cases:

a) The column is full and its radius is 50 cm

b) The column is hollow in the center: Its external radius is 50 cm and its internal radius is 49 cm.

Solution

Free body diagram

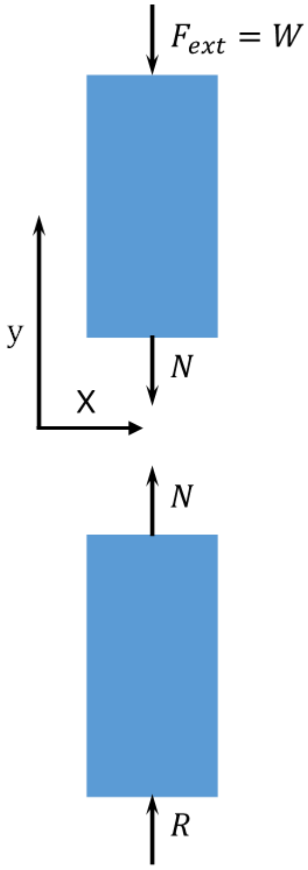

Diagram of the column showing the external force (weight of the statue), the reaction and the internal force in either side of the cut.

Objective:

a- $Lateral\ strain\ \varepsilon_{l_a}$

b- $Lateral\ strain\ \varepsilon_{l_b}$

Given:

m, mass of the statue

E, Young's modulus of the column material

ν, Poisson's ratio of the column material

H, Height of the statue (useless parameter)

a. r, radius of the cylinder

b. r_{out}, external radius of the cylinder, r_{int}, internal radius of the cylinder

Formulas:

$$F = m \cdot g \quad (1)$$

With F being the weight of the statue and g the gravitation constant

$$\sigma_y = \frac{N}{A} \tag{2}$$

Where A is the cross-section of the column that is given by:

$$A_a = \pi r^2; \quad A_b = \pi\left(r_{out}^2 - r_{in}^2\right) \tag{3}$$

Where r is the radius of the column in (a) and r_{out} and r_{in} are respectively the outer and inner radius of the column in (b).

Hooke's law in 1D

$$\sigma_y = E \cdot \varepsilon_y \tag{4}$$

where σ is the normal internal stress of the material, ε is the axial strain of the material and E is the Young's modulus of the material. And, finally, we have the axial-lateral strain relation, only valid when the load is uniaxial, i.e. $\sigma_y \neq 0, \sigma_x = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$

$$\varepsilon_x = \varepsilon_z = -\nu \cdot \varepsilon_y \tag{5}$$

For any of the three cases, we will have an equilibrium situation

$$N + F = 0 \rightarrow N = -F = -m \cdot g \tag{6}$$

Therefore, following this statement and combining with equation (2) we have

$$\sigma_y = -\frac{mg}{A} \tag{7}$$

We then apply Hooke's law

$$\sigma_y = E\varepsilon_y \rightarrow \varepsilon_y = \frac{\sigma_y}{E} = -\frac{mg}{EA} \tag{8}$$

And thus, we conclude that for any of the columns in the problem

$$\varepsilon_x = \varepsilon_z = \nu\frac{mg}{AE} \tag{9}$$

<u>Calculation</u>

The stress is compressive. We can now take into account case by case

a.

$$A_a = \pi r^2 \rightarrow \varepsilon_{x,z_a} = \frac{mg\nu}{\pi r^2 E} \tag{10}$$

b.

$$A_b = \pi\left(r_{out}^2 - r_{in}^2\right) \rightarrow \varepsilon_{x,z_b} = \frac{mg\nu}{\pi\left(r_{out}^2 - r_{in}^2\right)E} \tag{11}$$

<u>Numerical application</u>

a.

$$\varepsilon_{x,z_a} = 3.41 \cdot 10^{-6} \tag{12}$$

b.

$$\varepsilon_{x,z_b} = 8.61 \cdot 10^{-5} \tag{13}$$

Exercise 2b.2 – Hooke's Law in 3D

We apply a load F on a cubic piece of 1 mm width as shown in Figure 2b.2. The material's Young modulus is 200 GPa, Poisson's ratio 0.25 and the measured stress tensor is given as follow. We suppose the material to be homogeneous and isotropic. We worked at constant ambient temperature (20^{0}C).

$$\sigma = \begin{bmatrix} 100 & 30 & 80 \\ 30 & 180 & 50 \\ 80 & 50 & 220 \end{bmatrix} \text{MPa}$$

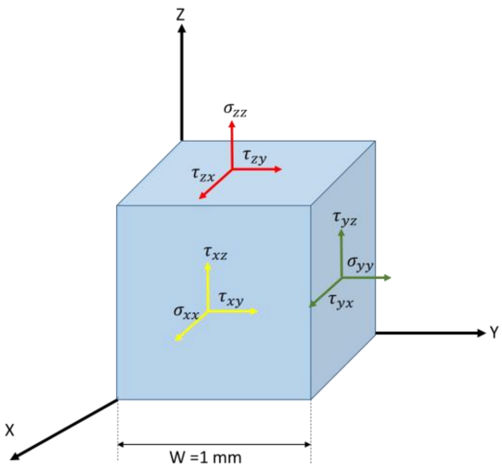

Figure 2b.2 | Stress components on a 3D cube

a) Determine the compliance matrix of the material
b) Determine the stress vector of this cube, when submitted to this load
c) Determine the strain vector of this cube, when submitted to this load
d) What has been the volume variation of the solid after the load has been applied?

Solution

a) Determine the compliance matrix of the material

We define the compliance matrix. Where ν is the Poisson's ratio of the material.

$$[C] = \frac{1}{E}\begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{pmatrix} \tag{14}$$

The numerical values of the compliance matrix are found by substituting the values of Poisson's ratio and Young's modulus and we get

$$[C] = \begin{pmatrix} .005 & -.00125 & -.00125 & 0 & 0 & 0 \\ -.00125 & .005 & -.00125 & 0 & 0 & 0 \\ -.00125 & -.00125 & .005 & 0 & 0 & 0 \\ 0 & 0 & 0 & .0125 & 0 & 0 \\ 0 & 0 & 0 & 0 & .0125 & 0 \\ 0 & 0 & 0 & 0 & 0 & .0125 \end{pmatrix} \text{GPa}^{-1} \tag{15}$$

b) Determine the stress vector of this cube, when submitted to this load

We start by the definition of stress tensor:

$$\hat{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{16}$$

The stress tensor can also be written in the form of a vector

$$\vec{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} \tag{17}$$

We take the given stress tensors and put it into vector form, to get the stress vector

$$\vec{\sigma} = \begin{pmatrix} 100 \\ 180 \\ 220 \\ 30 \\ 80 \\ 50 \end{pmatrix} \text{MPa} \tag{18}$$

c) Determine the strain vector of this cube, when submitted to this load

We start with the generalized Hooke's law

$$\vec{\varepsilon} = [C]\vec{\sigma} \tag{19}$$

Inserting into this equation the matrix from equation 18 and the tensor from equation 15 we can find the values of the strain vector

$$\vec{\varepsilon} = \begin{pmatrix} 0 \\ 0.5 \\ 0.75 \\ 0.375 \\ 1 \\ 0.625 \end{pmatrix} \cdot 10^{-3} \tag{20}$$

d) What has been the volume variation of the cube after the load has been applied?

We give the volume relative variation in correspondence to the stress parameters.

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1 - 2\nu}{\mathrm{E}}(\sigma_x + \sigma_y + \sigma_z) \tag{21}$$

Such that if we fill in either the strains calculated, or the given stresses, we find

$$\Delta V = V\frac{1 - 2\nu}{\mathrm{E}}\left(\sigma_x + \sigma_y + \sigma_z\right) \rightarrow \Delta V = 1.25 \cdot 10^{-12}\ \mathrm{m}^3 \tag{22}$$

Exercise 2b.3 – Poisson Ratio and Elasticity modulus

Consider a magnesium plate (seen in Figure 2b.3.1) in biaxial stress being subjected to the tensile stresses $\boldsymbol{\sigma_x}$ = 24 MPa and $\boldsymbol{\sigma_y}$ =12 MPa. The corresponding strains in the plate are $\varepsilon_x = 440\cdot10^{-6}$ and $\varepsilon_y = 80\cdot10^{-6}$.

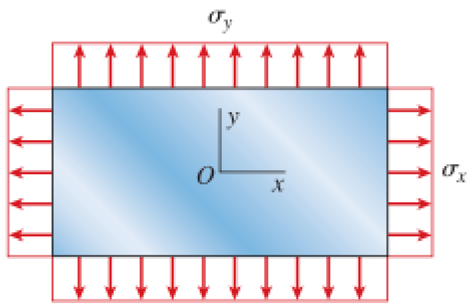

Figure 2b.3.1 | Block with Applied Stresses

a) Determine the Poisson's ratio and Young's modulus of the material

b) Determine the volume change in percentage of the material under these loads and strains

Suppose the magnesium plate is now fixed to the two fixed plates above and below, as shown in Figure 2b.3.2, which prevent any deformation in the z axis. When the plane-stresses are now applied a reaction force causing a stress of 12.6 MPa will occur on the body. Consider the same Poisson's ratio and Young's modulus

c) Determine how these constraints influences the volume change of the material

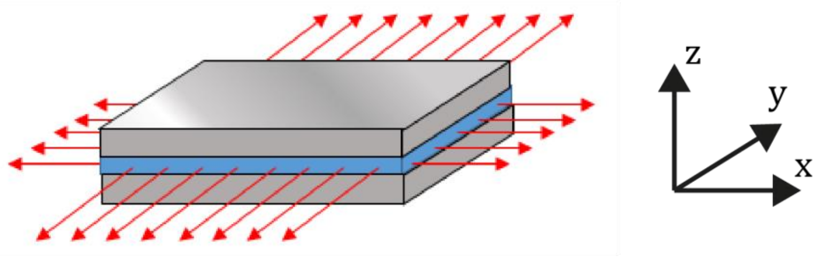

Figure 2b.3.2 | Block with constraints applied on the two sides of the plate

Solution

a) Determine the Poisson's ratio and Young's modulus of the material.

We start with the generalized Hooke's equations

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) \tag{1}$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right) \tag{2}$$

Since we know that $\sigma_x = 2\sigma_y$ and $\sigma_z = 0$ we can write

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \text{ and } \varepsilon_y = \frac{1}{E}(\sigma_y - 2\nu\sigma_y) \tag{3}$$

We can subtract the two parts from each other to eliminate ν

$$\varepsilon_x - \frac{1}{2}\varepsilon_y = \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{1}{2E}(\sigma_y - 2\nu\sigma_y) = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \frac{\sigma_y}{2} + \nu\sigma_y\right) \tag{4}$$

We can then rewrite this expression in terms of E and rewrite to solve

$$E = \frac{\sigma_x - \frac{1}{2}\sigma_y}{\varepsilon_x - \frac{1}{2}\varepsilon_y} = \frac{(24-6)*10^6}{(440-40)*10^{-6}} = 45 \text{ GPa} \tag{5}$$

Then we can rewrite either one of the formulas to find ν

$$E\varepsilon_x = \sigma_x - \nu\sigma_y \tag{6}$$

$$\nu = \frac{\sigma_x - E\varepsilon_x}{\sigma_y} \tag{7}$$

$$\nu = \frac{24*10^6 - 45*10^9*440*10^{-6}}{12*10^6} = 0.35 \tag{8}$$

b) Determine the volume change of the material in terms of its original volume V

The volume change of any linear elastic body is defined as

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{9}$$

Using the fact that the there's no stress in the z direction ($\sigma_z = 0$) we start with the three simplified strain components by means of the generalized Hooke's law

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{10}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{11}$$

$$\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y) \tag{12}$$

Which if we calculate these, we find

$$\varepsilon_x = \frac{(24 - 0.35 * 12) * 10^6}{45 * 10^9} = 4.4 * 10^{-4} \tag{13}$$

$$\varepsilon_y = \frac{(12 - 0.35 * 24) * 10^6}{45 * 10^9} = 0.8 * 10^{-4} \tag{14}$$

$$\varepsilon_z = -\frac{0.35}{45 * 10^9}(24 + 12) * 10^6 = -2.8 * 10^{-4} \tag{15}$$

Inserting this in the formula for the volume change we find that

$$\Delta V = (\varepsilon_x + \varepsilon_y + \varepsilon_z)V = 2.4 * 10^{-4} * V \tag{16}$$

This means that the body will change 0.024% in terms of volume.

c) Determine how these constraints influences the volume change of the material

With the plates in place the block can no longer expand in the z direction and thus the strain in this direction becomes zero ($\varepsilon_z = 0$). However, this constraints does ensure a counter force on the plate at a given value of σ_z = 12.6 MPa. To now calculate the volume change we need to recalculate the other two strain components

$$\varepsilon_x = \frac{1}{E}\{\sigma_x - \nu(\sigma_y + \sigma_z)\} \tag{17}$$

$$\varepsilon_y = \frac{1}{E}\{\sigma_y - \nu(\sigma_x + \sigma_z)\} \tag{18}$$

Which, with the provided information we find as

$$\varepsilon_x = \frac{\{24 - 0.35(12 + 12.6)\} * 10^6}{45 * 10^9} = 3.42 * 10^{-4} \tag{19}$$

$$\varepsilon_y = \frac{\{12 - 0.35(24 + 12.6)\} * 10^6}{45 * 10^9} = -1.8 * 10^{-5} \tag{20}$$

That means that the new volume change is

$$\Delta V' = (\varepsilon_x + \varepsilon_y)V = 3.24 * 10^{-4} * V \tag{21}$$

$$\Delta V' - \Delta V = 0.84 * 10^{-4} * V \tag{22}$$

The new volume change is 0.032% and thus the constraint increases the volume by an additional 0.0084%

Exercise 2b.4 – Pipe casing

A plastic cylindrical pipe is inserted as a liner inside a cast-iron pipe. We compress the plastic pipe with a load F. We use the parameters listed in the following table and represented in Figure 2b.4.

Parameters	Plastic pipe	Cast-iron pipe
Length	$L_p = ?$	$L_c = 0.205\ m$
Diameter	$d_1 = 109\ mm$	Inner : $d_2 = 110\ mm$ Outer : $d_3 = 115\ mm$
Young Modulus	$E_p = 2.1\ GPa$	$E_c = 170\ GPa$
Poisson's ratio	$\nu_p = 0.4$	$\nu_c = 0.3$

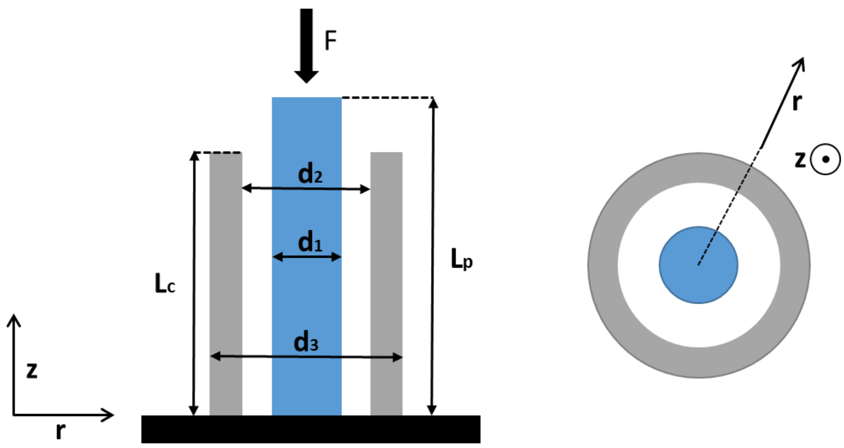

Figure 2b.4 | Plastic pipe in cast-iron pipe

We compress the plastic pipe with F so that the final length of both pipes is the same and also, at the same time, the final outer diameter of the plastic pipe is equal to the inner diameter of the cast-iron pipe so that no stress is applied on the cast-iron pipe.

a) Derive a formula for the required initial length L_p with the Poisson's ratio (not as a function of F). Calculate the value of L_p.

b) What is the final stress along in the vertical and the radial direction in the plastic and in the cast-iron pipe?

c) What is the required force F?

d) Determine the ratio between final and initial volumes for the plastic pipe.

Solution

a) Derive a formula for the required initial length L_p with the Poisson's ratio (not as a function of *F*). Calculate the value of L_p.

The lateral strain resulting from compression of the plastic pipe must close the gap between it and the cast-iron one. Therefore, we consider that the lateral displacement for this calculation covers the distance $d_2 - d_1$, ie 1 mm. We call it Δd.

The increase in the outer diameter of the plastic pipe leads to an increase in the radius to accommodate an increase in the circumference length. Therefore, we can write for the final diameter of the plastic pipe that:

$$d_{1,final} = d_1 \cdot (1 + \varepsilon_r) = d_2 \tag{1}$$

The accompanying compressive normal stress in the plastic pipe is obtained using the Poisson's ratio formula which describes the relation between compressive and radial strain.

$$\nu_p = -\frac{\varepsilon_{p,r}}{\varepsilon_{p,z}} \rightarrow \varepsilon_{p,z} = -\frac{\varepsilon_{p,r}}{\nu_p} = -\frac{\Delta d}{\nu_p d_1} \tag{2}$$

Now, by definition, we determine the axial deformation of the plastic pipe.

$$\delta_p = \varepsilon_{p,z} L_p \tag{3}$$

Knowing also that

$$\delta_p = L_c - L_p \tag{4}$$

Equalizing both equations, we get

$$\varepsilon_{p,z} L_p = L_c - L_p \rightarrow L_p = \frac{L_c}{1 + \varepsilon_{p,z}} \tag{5}$$

By substituting the strain, we get and we find

$$L_p = \frac{L_c}{1 - \frac{\Delta d}{\nu_p d_1}} = \frac{0.205[m]}{1 - \frac{110 - 109[mm]}{0.4 * 109[mm]}} = 0.210m \tag{6}$$

b) What is the final stress along in the vertical and the radial direction in the plastic and in the cast-iron pipe?

For the cast-iron pipe, there is no stress since it is not affected by the load.

$$\sigma_{c,z} = \sigma_{c,r} = \sigma_{p,r} = 0\ Pa \tag{7}$$

For the plastic pipe, there is only stress in the axial direction since it is extended freely laterally until it meets the wall. The normal stress in the plastic pipe is thus calculated thanks to Hooke's law in the axial direction

$$\varepsilon_{p,z} = -\frac{\varepsilon_{p,r}}{\nu_p}\ or\ \varepsilon_{p,z} = \frac{\Delta L}{L_p} = \frac{L_c - L_p}{L_p} \tag{8}$$

$$\sigma_{p,z} = E_p \varepsilon_{p,z} = -E_p \frac{\Delta d}{\nu_p d_1} \tag{9}$$

With E_p being the Young modulus of the material that composes the plastic pipe. Finally filling in the values we find the value of the normal stress in the pipe.

$$\sigma_{p,z} = -E_p \frac{\Delta d}{\nu_p d_1} = -48 \text{ MPa}$$
$$\text{or } \sigma_{p,z} = E_p \frac{\Delta L}{L_p} = 2.1[Gpa] * \frac{0.205 - 0.210[m]}{0.210[m]} = -50.0 \text{ MPa} \tag{10}$$

(small rouding error depending on precision of value L_p used)

c) What is the required force F?

We start by the definition of the axial strain

$$\varepsilon_{p,z} = \frac{\Delta L}{L_0} \tag{11}$$

where L_0 is the initial length of the zone studied, ΔL is the deformation the pipe is enduring, with $\varepsilon_{p,z}$ being the normal strain when the load is acting. As in our problem we only have stress in the axial direction (z) we can use the 1-dimensional form of the Hooke's Law

$$\sigma_{p,z} = E_p \varepsilon_{p,z} \tag{12}$$

To calculate the strain in the radial direction we again use Poisson's ratio

$$\varepsilon_{p,r} = -\nu_p \cdot \varepsilon_{p,z} = -\frac{\nu_p}{E_p} \sigma_{p,z} \tag{13}$$

where ε_r is the substructure lateral strain when the load is acting, ε_z is the pipe axial strain when the load is acting, ν_p is the Poisson's ratio of the pipe's material. We know that we can define the applied stress and rewrite to find the applied load F

$$\sigma_{p,z} = \frac{F}{A_p} \rightarrow= F = A_p \sigma_{p,z} = \frac{\pi {d_1}^2}{4} \sigma_{p,z} \tag{14}$$

The required downwards load to compress the plastic pipe as wanted is thus determined by

$$F = \sigma_{p,z} A_p = E_p \varepsilon_{p,z} A_p = -E_p \frac{\Delta d}{\nu_p d_1} \frac{\pi d_1^2}{4} = -E_p \frac{\Delta d}{\nu_p} \frac{\pi d_1}{4}$$
$$or\ F = E_p \frac{\Delta L}{L_p} \frac{\pi d_1^2}{4} \tag{15}$$

Finally, by plugging in the numbers we find the value of the applied force

$$F = -2.1[\text{Gpa}] * \frac{110 - 109[mm]}{0.4} * \frac{\pi * 109[\text{mm}]}{4} = -449.4kN$$
$$or\ F = 2.1[Gpa] * \frac{0.205 - 0.210}{0.210} * \frac{\pi * 109[\text{mm}] * 109[\text{mm}]}{4} = -466.6kN \tag{16}$$

(small rouding error depending on precision of value L_p used)

F is negative, therefore facing downwards and thus compressive.

d) Determine the ratio between final and initial volumes for the plastic pipe.

The initial volume and final volume of the pipe are given by

$$V_{p_{initial}} = L_p A_p$$
$$V_{p_{final}} = L_c A_{\mathrm{P},final} \tag{17}$$

The ratio of the final to initial volume is therefore

$$\frac{V_{p_{final}}}{V_{p_{initial}}} = \frac{L_c A_{p_{final}}}{L_p A_p} \tag{18}$$

$$\frac{V_{p_{final}}}{V_{p_{initial}}} = \frac{0.205[\mathrm{m}] * \frac{\pi * 110[mm] * 110[mm]}{4}}{0.210[\mathrm{m}] * \frac{\pi * 109[mm] * 109[mm]}{4}} = 0.995 \tag{19}$$

Exercise 2b.5 – Dielectric actuator design

A Dielectric Actuator is an actuator which finds it's working principle in an applied electric field between two stretchable electrodes. As a voltage is applied, the two electrodes are attracted to each other through electrostatic attraction. A dielectric film prevents the flow of current.

The electrostatic force within the dielectric, under an applied electric field, is given as

$$F_z = -\frac{1}{2}\frac{{Q_p}^2}{\varepsilon_d A_p}$$

Where A_p is the planar cross-sectional area (x-y plane) and ε_d the dielectric permittivity. The charges in the plane are given by

$$Q_p = \frac{V\varepsilon_d A_p}{d_0}$$

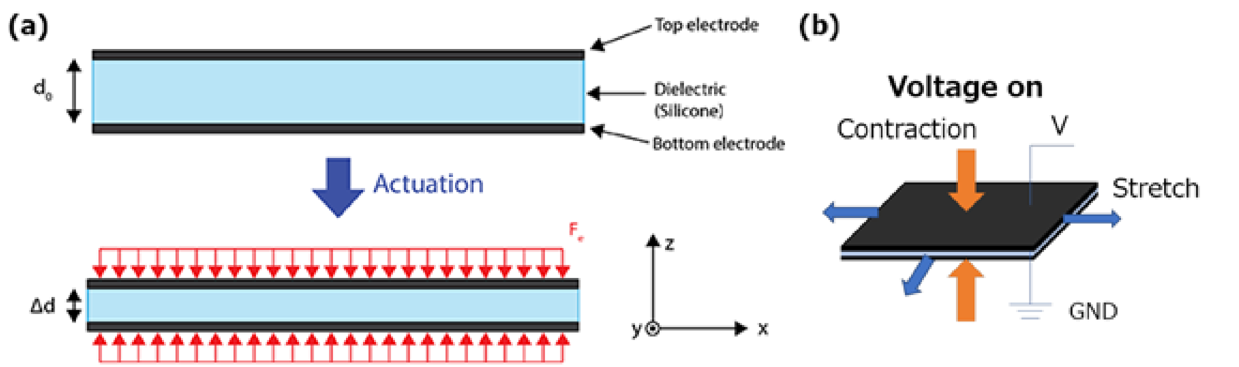

Figure 2b.5 | (a) Sketch of the problem, **(b)** working principle of a Dielectric Actuator

Where d_0 stands for the dielectric thickness and V is the applied voltage. When the dielectric is squeezed by the applied voltage, it also expands in-plane. In elastomers the compression and the expansion are mechanically coupled due to their incompressibility. The tensile stress due to this expansion equals to the compressive stress (this is one way to find the stress components):

$$\sigma_x = \sigma_y = -\sigma_z$$

Now suppose we have the actuator, as shown in figure 2.b.5, with a 200 µm thick dielectric made out of silicone which we actuate this under a voltage of 1 [kV]. The dielectric permittivity is 35.4*10^{-12} [N/V^2], and the material has a Poisson's ratio of 0.5, as well as an elastic modulus of 1.2 [MPa]. The material behaves isotropic and has linear elastic properties. We ignore the stiffness and any influence of the top electrodes.

a) **Under the given load, calculate the tensile and compressive stresses in the dielectric**
b) **For the calculated stresses, find the strain components of the dielectric**
c) **Give the change in volume of the dielectric**
d) **Suppose we introduce an additional shear stress of 1 kPa in the positive x direction on the x-y plane. What effect does this have on the already present strain state and change in volume?**

Solution

a) Under the given load, calculate the stress in the dielectric

We know that we can safely ignore the stiffness of the top electrodes, the forces given by the problem are as per

$$\mathrm{F}_z = -\frac{1}{2}\frac{{Q_p}^2}{\varepsilon_d A_p} \tag{1}$$

We then insert the definitions of respective charges into equation 1 to get the definitions of the forces

$$\mathrm{F}_z = -\frac{1}{2}\left(\frac{V^2{\varepsilon_d}^2{A_p}^2}{\varepsilon_d {d_0}^2 A_p}\right) = -\frac{1}{2}\varepsilon_d A_p\left(\frac{V}{d_0}\right)^2 \tag{2}$$

Then dividing the force by its respective surface, the tensile stress in the dielectric will be equal to

$$\sigma_z = \frac{\mathrm{F}_z}{A_p} \tag{3}$$

Where if we insert the definition of the force inside of equation 3, we will retrieve the tensile Maxwell Stresses.

$$\sigma_z = -\frac{1}{2}\varepsilon_d\left(\frac{V}{d_0}\right)^2 \tag{4}$$

And then resulting from the given stress relation we find

$$\sigma_x = \sigma_y = -\sigma_z = \frac{1}{2}\varepsilon_d\left(\frac{V}{d_0}\right)^2 \tag{5}$$

This differs from what we are used to, as generally in mechanics a vertical load doesn't cause stress in the other directions! However, since we're dealing with electrostatics these kinds of mechanical forces are the result of the electrical charges.

With the provided values we can thus calculate the stresses present in the dielectric.

$$\begin{gathered}\sigma_z = -\frac{1}{2}\varepsilon_d\left(\frac{1000}{200 * 10^{-6}}\right)^2 \approx -442.25\ [\mathrm{Pa}] \\ \sigma_x = \sigma_y = -\sigma_z = 442.25\ [\mathrm{Pa}]\end{gathered} \tag{6}$$

b) For the calculated stress, find the strain components of the dielectric

Starting from the Compliance matrix for isotropic materials, and knowing the relations between the stresses are provided as in equation 3, we can simplify it and retrieve the strain equations

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix}\begin{pmatrix} \sigma \\ \sigma \\ -\sigma \end{pmatrix} \tag{7}$$

$$\begin{gathered} \varepsilon_x = \varepsilon_y = \frac{1}{E}(\sigma - \nu\sigma + \nu\sigma) = \frac{\sigma}{E} \\ \varepsilon_z = \frac{1}{E}(-\nu\sigma - \nu\sigma - \sigma) = -\frac{\sigma}{E}(2\nu + 1) \end{gathered} \tag{8}$$

Which we can then calculate (pay attention to the signs!)

$$\begin{gathered} \varepsilon_x, \varepsilon_y = \frac{442.25}{1.2 * 10^6} \approx 3.685 * 10^{-4} \\ \varepsilon_z = -\frac{442.25}{1.2 * 10^6} * (2 * 0.5 + 1) \approx -7.370 * 10^{-4} \end{gathered} \tag{9}$$

c) Give the change in volume of the dielectric

As the Poisson ratio is 0.5, the dielectric is <u>incompressible</u>. Meaning that the volume change is zero. We can show this by looking at the change in volume

$$\frac{\Delta V}{V} = \{\varepsilon_x + \varepsilon_y + \varepsilon_z\} \tag{10}$$

$$\Delta V = \{3.685 + 3.685 - 7.370\} * 10^{-4} * V = 0 \tag{11}$$

Zero volume change, as stated before, the film is incompressible!

d) Suppose we introduce an additional shear stress of 1 kPa in the positive x direction on the x-y plane. What effect does this have on the already present strain state and change in volume?

For this last part we have to look at the compliance matrix first which is given as

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{pmatrix}\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{pmatrix} \tag{12}$$

As we can see, because the system is linear and isotropic, the normal and shear strains are decoupled and there will be no change for the compression in the z direction. We do however generate a shear strain in the x-y direction. First, we calculate the shear modulus

$$G = \frac{E}{2(1+\nu)} = \frac{1.2 * 10^6}{2(1+0.5)} = 4 * 10^5 \text{ [Pa]} = 400 \text{ [kPa]} \tag{13}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{1000}{4 * 10^5} = 25 * 10^{-4} \tag{14}$$

In an elastic isotropic system, the shear strain doesn't have an influence on the volume change, so it remains zero.

